PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 22, Fasc. 1 (2002), pp. 167–180

STOPPING AND STOCHASTIC INTEGRALS AS CLOSABLE OPERATORS

BY

ABDULRAHMAN A. A. MOHAMMED (ŁÓDŹ)

Abstract. We show that for a von Neumann algebra in standard form with cyclic and separating tracial vector, and some classes of noncommutative processes on it, stopping and integrals of these processes can be treated as closable operators whose closures are affiliated to the algebra.

1. INTRODUCTION

Stopping of noncommutative processes was studied in [1], [2], [4], [6] under various circumstances; however, a common feature of all these approaches was the following: for a process $(X(t))$ and a random time τ, stopping $(X(t))$ by τ, X_τ, is an element of some Hilbert space on which the von Neumann algebra under consideration acts. In this paper we present another point of view on stopping as well as on stochastic integrals – namely, we shall show that they can be treated as closable operators whose closures are affiliated to the algebra. For stochastic integrals in quasi-free representations of the CAR and CCR algebras this approach was considered in [3], with the integrator being the 'canonical' CAR or CCR martingale. It turns out that in general both stopping and integration can be looked upon in this way if we restrict our attention to the class $\mathscr{U}(\mathscr{P})$ of predictable processes and the von Neumann algebra in standard form with cyclic and separating tracial vector.

1. PRELIMINARIES AND NOTATION

A noncommutative stochastic base which we shall be working in consists of the following elements: a von Neumann algebra $\mathscr{A}$ acting on a Hilbert space $\mathscr{H}$, a normal faithful trace φ on $\mathscr{A}$, a filtration $(\mathscr{A}_t)$, $t \in [0, +\infty]$, which is an increasing ($s \leqslant t$ implies $\mathscr{A}_s \subset \mathscr{A}_t$) family of von Neumann subalgebras of $\mathscr{A}$ such that $\mathscr{A} = \mathscr{A}_\infty = \left(\bigcup_{t \geqslant 0} \mathscr{A}_t\right)''$ and $\mathscr{A}_s = \bigcap_{t > s} \mathscr{A}_t$ (right-continuity). Then for each $t \geqslant 0$ there exists a normal conditional expectation M_t from $\mathscr{A}$ onto

$\mathscr{A}_t$ such that $\varphi \circ M_t = \varphi$. We shall assume that there is a cyclic and separating unit vector $\Omega \in \mathscr{H}$ such that $\varphi(a) = \langle a\Omega, \Omega \rangle$, $a \in \mathscr{A}$.

For each $t \in [0, +\infty]$ we write $L^2(\mathscr{A}_t)$ for the noncommutative Lebesgue space associated with $\mathscr{A}_t$ and φ. The theory of these spaces is described e.g. in [9]; for our purposes we recall only that $L^2(\mathscr{A})$ (accordingly $L^2(\mathscr{A}_t)$) consists of densely defined operators on $\mathscr{H}$, affiliated to $\mathscr{A}$, and that $L^2(\mathscr{A})$ is the completion of $\mathscr{A}$ with respect to the norm

$$\|X\|_2 = [\varphi(|X|^2)]^{1/2};$$

moreover, for $a \in \mathscr{A}$ and $X \in L^2(\mathscr{A})$, the operators aX and Xa belong to $L^2(\mathscr{A})$. For each t the conditional expectation M_t extends to the projection from $L^2(\mathscr{A})$ onto $L^2(\mathscr{A}_t)$.

By an *$\mathscr{A}$-valued* (respectively, *L^2-valued*) *process* we mean a map from $[0, +\infty]$ into $\mathscr{A}$ (respectively, $L^2(\mathscr{A})$). A process $(X(t))$ is called *adapted* if $X(t) \in \mathscr{A}_t$ ($X(t) \in L^2(\mathscr{A}_t)$, respectively).

Let us introduce the notion of a random time.

DEFINITION 1.1. A *random time* is a map τ: $[0, +\infty] \to \operatorname{Proj} \mathscr{A}$ such that $\tau(0) = 0$, $\tau(+\infty) = \mathbf{1}$, $\tau(t)$ is a projection in $\mathscr{A}_t$, and $\tau(s) \leqslant \tau(t)$ whenever $s \leqslant t$.

This definition is adopted from [2], [4], [5]. Random times will often be denoted by $\tau = (E_t)$, which means that $\tau(t) = E_t$. A random time is called *simple* if it assumes only finitely many values.

For random times there is a partial ordering, namely, we mean that $\sigma \leqslant \tau$, $\sigma = (F_t)$, $\tau = (E_t)$, if $E_t \leqslant F_t$ for each $t \in [0, +\infty]$.

2. STOPPING AS A CLOSABLE OPERATOR

Let us recall the following definitions from [6].

DEFINITION 2.1. (i) Let $\sigma = (F_t)$ and $\tau = (E_t)$ be random times with $\sigma \leqslant \tau$. The *stochastic interval* $(\sigma, \tau]$ is a process defined as

$$(\sigma, \tau](t) = F_t - E_t, \qquad t \in [0, +\infty].$$

(ii) For a projection P in $\mathscr{A}_0$ we define *'interval'* $[0_P]$ as

$$[0_P](t) = \begin{cases} P & \text{for } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION 2.2. (i) Let λ_i for $i = 0, 1, \ldots, n$ be complex numbers, P a projection in $\mathscr{A}_0$, and $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n$ random times with $\sigma_1 \leqslant \tau_1 \leqslant \sigma_2 \leqslant \leqslant \tau_2 \leqslant \ldots \leqslant \sigma_n \leqslant \tau_n$. Any process f of the form

$$f = \lambda_0 [0_P] + \sum_{i=1}^{n} \lambda_i (\sigma_i, \tau_i]$$

is called an *elementary predictable process.*

(ii) A process which is a finite linear combination of finite products of elementary predictable processes is called a *simple predictable process.*

Note that since elementary predictable processes are clearly adapted, a simple predictable process is also adapted. In fact, the restriction that $\tau_1 \leqslant \sigma_2 \leqslant \ldots \leqslant \tau_{n-1} \leqslant \sigma_n$ is inessential in our further considerations (as well as in the results of [6]), so as an elementary predictable process we can take any linear combination of stochastic intervals and the process $[0_P]$. In the same manner we can extend the definition of a simple predictable process admitting elementary predictable processes in the above sense.

We introduce another important class of processes.

DEFINITION 2.3. We define $\mathscr{U}(\mathscr{P})$ to be the class of those processes which are the uniform limits of sequences of simple predictable processes each of which is a finite linear combination of elementary predictable processes (note that we exclude products of elementary predictable processes).

Thus $f \in \mathscr{U}(\mathscr{P})$ if there exists a sequence $(f^{(n)})$ of simple predictable processes such that $f^{(n)}$ is a finite linear combination of elementary processes and

$$\lim_{n\to\infty} \sup_t \|f^{(n)}(t) - f(t)\|_\infty = 0.$$

In [6] the analysis of stopping of processes from $\mathscr{U}(\mathscr{P})$ is performed. The present work is devoted to the same subject but in a different setting, which leads to the notions of stopping and integral as densely defined operators. Let us recall the notion of stopping. For an $\mathscr{A}$-valued process $(f(t))$, a random time $\tau = (E_t)$ and a partition $\theta = \{0 = t_0 < t_1 < \ldots < t_m = +\infty\}$ of $[0, +\infty]$ we put

$$f_{\tau(\theta)} = \sum_{i=1}^{m} f(t_i)(E_{t_i} - E_{t_{i-1}}).$$

If there exists $\lim_\theta f_{\tau(\theta)}$ as θ refines, it is denoted by f_τ and called *stopping the process* $(f(t))$ *by the random time* τ (see [1], [2], [5], [6] for motivations and comments). The above limit is almost exclusively considered in the L^2-norm, and accordingly f_τ is an element of $L^2(\mathscr{A})$. Our approach will be different. Namely, $f_{\tau(\theta)}$ is an operator in $\mathscr{A}$, and we may ask about the existence of $\lim_\theta f_{\tau(\theta)}\Omega$. If this limit does exists,

$$\lim_\theta f_{\tau(\theta)}\Omega = \xi, \tag{2.1}$$

then, taking into account the relation

$$f_{\tau(\theta)} a' = a' f_{\tau(\theta)}, \quad a' \in \mathscr{A}',$$

we may define f_τ by the formula

(2.2) $$f_\tau(a'\Omega) = a'\xi, \qquad a' \in \mathscr{A}'.$$

It follows that f_τ is a densely defined linear operator on $\mathscr{H}$. Moreover, we have

$$f_\tau(a'\Omega) = a'(\lim_\theta f_{\tau(\theta)}\Omega) = \lim_\theta a' f_{\tau(\theta)}\Omega = \lim_\theta f_{\tau(\theta)}(a'\Omega).$$

Analogously, considering sums

$$_{\tau(\theta)}f = \sum_{i=1}^{m} (E_{t_i} - E_{t_{i-1}}) f(t_i)$$

and assuming the existence of the limit

$$\lim_\theta {}_{\tau(\theta)} f\Omega = \eta,$$

we obtain a densely defined linear operator $_\tau f$ on $\mathscr{H}$:

$$_\tau f(a'\Omega) = a'\eta, \qquad a' \in \mathscr{A}'.$$

We shall call f_τ and $_\tau f$ the *right* and *left stopping* of $(f(t))$, respectively. In what follows we formulate our results for right stopping, their 'left' counterparts being obvious. The basic properties of stopping are given by the following theorem.

THEOREM 2.4. *Let $(f(t))$ be an $\mathscr{A}$-valued process and let $\tau = (E_t)$ be a random time. Assume that the limit* (2.1) *exists, and let f_τ be defined by* (2.2). *Then f_τ is closable and its closure is affiliated to $\mathscr{A}$. Moreover, there exists left stopping of the adjoint process $(f(t)^*)$, $_\tau(f^*)$, and we have*

$$\bar{f_\tau} = [_\tau(f^*)]^*.$$

Proof. For any partitions θ', θ'' we have

$$\|f_{\tau(\theta')}\Omega - f_{\tau(\theta'')}\Omega\| = \|f^*_{\tau(\theta')}\Omega - f^*_{\tau(\theta'')}\Omega\|,$$

which means that the net $\{f^*_{\tau(\theta)}\Omega\}$ is Cauchy, since such is the net $\{f_{\tau(\theta)}\Omega\}$. Thus there exists

$$\lim_\theta f^*_{\tau(\theta)}\Omega = \eta.$$

Since

$$f^*_{\tau(\theta)} = {}_{\tau(\theta)}(f^*),$$

where the symbol $_{\tau(\theta)}(f^*)$ means that we are dealing with left stopping of the process $(f(t)^*)$, we may define an operator $_\tau(f^*)$ by

$$_\tau(f^*)(a'\Omega) = a'\eta, \qquad a' \in \mathscr{A}'.$$

It is a densely defined linear operator; moreover, for $a', b' \in \mathscr{A}'$ we have

$$\langle f_\tau(a'\Omega), b'\Omega\rangle = \lim_\theta \langle f_{\tau(\theta)}(a'\Omega), b'\Omega\rangle = \lim_\theta \langle a'\Omega, f^*_{\tau(\theta)}(b'\Omega)\rangle$$

$$= \lim_\theta \langle a'\Omega, {}_{\tau(\theta)}(f^*)(b'\Omega)\rangle = \langle a'\Omega, {}_\tau(f^*)(b'\Omega)\rangle,$$

which yields the inclusion

$$f_\tau \subset [{}_\tau(f^*)]^*.$$

So f_τ is closable and

$$\bar{f}_\tau \subset [{}_\tau(f^*)]^*.$$

Since

$$(b' f_\tau)(a'\Omega) = b'\big(\lim_\theta f_{\tau(\theta)}(a'\Omega)\big)$$

$$= \lim_\theta f_{\tau(\theta)}(b'a'\Omega) = (f_\tau b')(a'\Omega), \qquad a', b' \in \mathscr{A}',$$

affiliation of f_τ, and thus that of $\bar{f}_\tau$ follows. Similarly, ${}_\tau(f^*)$, and hence $[{}_\tau(f^*)]^*$ is affiliated to $\mathscr{A}$. Now $\mathscr{A}$ is a finite von Neumann algebra, and $\bar{f}_\tau$ and $[{}_\tau(f^*)]^*$ are closed linear operators affiliated to $\mathscr{A}$ such that $\bar{f}_\tau \subset [{}_\tau(f^*)]^*$. Thus the equality $\bar{f}_\tau = [{}_\tau(f^*)]^*$ follows from Theorem 9.8 of [8]. ∎

In what follows, to avoid repeating the phrase 'f_τ is closable with its closure affiliated to $\mathscr{A}$' we shall say that *there exists an operator stopping* f_τ.

COROLLARY 2.5. *If $f_\tau\Omega \in \mathscr{A}\Omega$, then $\bar{f}_\tau \in \mathscr{A}$.*

Indeed, if $f_\tau\Omega = a\Omega$ for some $a \in \mathscr{A}$, then for each $a' \in \mathscr{A}'$ we have

$$f_\tau(a'\Omega) = a' f_\tau \Omega = a'a\Omega = a(a'\Omega),$$

which means that $f_\tau = a$ on the dense subspace $\mathscr{A}'\Omega$, so $\bar{f}_\tau = a$.

Our aim in this section is to show that this form of stopping can be applied to predictable processes from $\mathscr{U}(\mathscr{P})$. First we show that we can stop any random time.

PROPOSITION 2.6. *Let $\sigma = (F_t)$ and $\tau = (E_t)$ be random times. Then there exists an operator stopping σ_τ.*

Proof. According to Theorem 2.4 all we need to show is the convergence of the net $\{\sigma_{\tau(\theta)}\Omega\}$, which on the other hand is essentially the result of Theorem 4.10 in [6]. Indeed, in that theorem it is shown that the net $\{\sigma_{\tau(\theta)}\Omega\}$ converges in L^2-norm, which by virtue of the equality

$$\|x\|_2^2 = \varphi(x^* x) = \langle x^* x\Omega, \Omega\rangle = \|x\Omega\|^2, \qquad x \in \mathscr{A},$$

shows the convergence of $\{\sigma_{\tau(\theta)}\Omega\}$. ∎

As immediate corollaries we obtain

Corollary 2.7. *Let f be an elementary predictable process, and let τ be a random time. Then there exists an operator stopping f_τ.*

Indeed, f is of the form

$$f = \lambda_0 [0_P] + \sum_{k=1}^{r} \lambda_k (\sigma_k, \varrho_k],$$

where $\sigma_k = (F_t^{(k)})$ and $\varrho_k = (Q_t^{(k)})$ are random times, so

$$f(t) = \lambda_0 [0_P](t) + \sum_{k=1}^{r} \lambda_k [F_t^{(k)} - Q_t^{(k)}]$$

and, consequently,

$$f_{\tau(\theta)} = \sum_{k=1}^{r} \lambda_k [\sigma_{\tau(\theta)}^{(k)} - \varrho_{\tau(\theta)}^{(k)}]$$

showing, by virtue of Proposition 2.6, the existence of $\lim_\theta f_{\tau(\theta)} \Omega$.

Corollary 2.8. *Let f be a simple predictable process which is a linear combination of elementary predictable processes, and let τ be a random time. Then there exists an operator stopping f_τ.*

This follows from the preceding corollary and the additivity of limit.

Before proving the main result of this section we need a version of 'contraction lemma' relating the Hilbert space norm of simple stopping with the operator norm of the stopped process.

Lemma 2.9. *Let $(f(t))$ be an $\mathscr{A}$-valued process, let $\tau = (E_t)$ be a random time, and let $\theta = \{0 = t_0 < t_1 < \ldots < t_m = +\infty\}$ be a partition of $[0, +\infty]$. Then*

$$\|f_{\tau(\theta)} \Omega\| \leqslant \sup_t \|f(t)\|_\infty .$$

Proof. We have

$$\|f_{\tau(\theta)} \Omega\|^2 = \Big\langle \sum_{i=1}^{m} f(t_i)(E_{t_i} - E_{t_{i-1}}) \Omega, \sum_{j=1}^{m} f(t_j)(E_{t_j} - E_{t_{j-1}}) \Omega \Big\rangle$$

$$= \sum_{i,j=1}^{m} \langle (E_{t_j} - E_{t_{j-1}}) f(t_j)^* f(t_i)(E_{t_i} - E_{t_{i-1}}) \Omega, \Omega \rangle$$

$$= \sum_{i,j=1}^{m} \langle f(t_j)^* f(t_i)(E_{t_i} - E_{t_{i-1}})(E_{t_j} - E_{t_{j-1}}) \Omega, \Omega \rangle$$

since Ω is tracial. Furthermore, the orthogonality of projections $E_{t_i}-E_{t_{i-1}}$ and $E_{t_j}-E_{t_{j-1}}$ for $i \neq j$ yields the formula

$$\begin{aligned}\|f_{\tau(\theta)}\Omega\|^2 &= \sum_{i=1}^{m} \langle f(t_i)^* f(t_i)(E_{t_i}-E_{t_{i-1}})\Omega, \Omega\rangle \\ &\leqslant \sum_{i=1}^{m} \|f(t_i)^* f(t_i)\|_\infty \langle (E_{t_i}-E_{t_{i-1}})\Omega, \Omega\rangle \\ &\leqslant \sup_t \|f(t)\|_\infty^2 \sum_{i=1}^{m} \langle (E_{t_i}-E_{t_{i-1}})\Omega, \Omega\rangle = \sup_t \|f(t)\|_\infty^2,\end{aligned}$$

which gives the claim. ∎

Now we are in a position to show the possibility of operator stopping of predictable processes in $\mathscr{U}(\mathscr{P})$. Namely, we have

THEOREM 2.10. *Let $f \in \mathscr{U}(\mathscr{P})$ and let τ be a random time. Then there exists an operator stopping f_τ.*

Proof. Let (f^n) be an approximating sequence for f of simple predictable processes which are finite linear combinations of elementary predictable processes, i.e.

$$\lim_{n\to\infty} \sup_t \|f(t)-f^{(n)}(t)\|_\infty = 0.$$

For given $\varepsilon > 0$ choose n_0 such that

$$\sup_t \|f(t)-f^{(n_0)}(t)\|_\infty < \varepsilon/4.$$

Since, by Corollary 2.8,

$$\lim_\theta f^{(n_0)}_{\tau(\theta)}\Omega = f^{(n_0)}_\tau \Omega,$$

we can find a partition θ_0 such that for each partition $\theta \geqslant \theta_0$

$$\|f^{(n_0)}_{\tau(\theta)}\Omega - f^{(n_0)}_\tau\Omega\| < \varepsilon/4. \tag{2.3}$$

For any partitions $\theta', \theta'' \geqslant \theta_0$ we then have

$$\begin{aligned}\|f_{\tau(\theta')}\Omega - f_{\tau(\theta'')}\Omega\| &\leqslant \|f_{\tau(\theta')}\Omega - f^{(n_0)}_{\tau(\theta')}\Omega\| \\ &\quad + \|f^{(n_0)}_{\tau(\theta')}\Omega - f^{(n_0)}_{\tau(\theta'')}\Omega\| + \|f^{(n_0)}_{\tau(\theta'')}\Omega - f_{\tau(\theta'')}\Omega\| \\ &\leqslant \|[f-f^{(n_0)}]_{\tau(\theta')}\Omega\| + \|f^{(n_0)}_{\tau(\theta')}\Omega - f^{(n_0)}_\tau\Omega\| + \|f^{(n_0)}_\tau\Omega - f^{(n_0)}_{\tau(\theta'')}\Omega\| + \|[f^{(n_0)}-f]_{\tau(\theta'')}\Omega\|.\end{aligned} \tag{2.4}$$

Now by Lemma 2.9 we get

$$\|[f-f^{(n_0)}]_{\tau(\theta')}\Omega\| \leqslant \sup_t \|f(t)-f^{(n_0)}(t)\|_\infty < \varepsilon/4,$$

$$\|[f^{(n_0)}-f]_{\tau(\theta'')}\Omega\| \leqslant \sup_t \|f^{(n_0)}(t)-f(t)\|_\infty < \varepsilon/4,$$

while the second and third terms on the right-hand side of the inequality (2.4) are, by (2.3), less than $\varepsilon/4$. This yields the inequality

$$\|f_{\tau(\theta')}\Omega - f_{\tau(\theta'')}\Omega\| < \varepsilon \quad \text{for } \theta', \theta'' \geqslant \theta_0,$$

which means that the net $\{f_{\tau(\theta)}\Omega\}$ is Cauchy, so it converges, proving in accordance with Theorem 2.4 the existence of operator stopping f_τ. ∎

We finish this section showing that f_τ can also be obtained as a limit of $f_\tau^{(n)}$, namely we have

THEOREM 2.11. *Let $f \in \mathscr{U}(\mathscr{P})$ and let $(f^{(n)})$ be an approximating sequence for f of simple predictable processes which are finite linear combinations of elementary predictable processes. Then*

$$\lim_{n\to\infty} f_\tau^{(n)}(a'\Omega) = f_\tau(a'\Omega), \qquad a' \in \mathscr{A}'.$$

Proof. First we shall show that $\lim_{n\to\infty} f_\tau^{(n)}\Omega = f_\tau\Omega$. Let us assume the contrary, i.e. that there exists a subsequence $\{n_k\}$ of positive integers such that

$$\|f_\tau^{(n_k)}\Omega - f_\tau\Omega\| \geqslant \varepsilon_0$$

for some $\varepsilon_0 > 0$. Choose k_0 large enough such that

$$\sup_t \|f^{(n_{k_0})}(t) - f(t)\| < \varepsilon_0/3.$$

Having chosen k_0 let us find a partition θ such that

$$\|f_{\tau(\theta)}^{(n_{k_0})}\Omega - f_\tau^{(n_{k_0})}\Omega\| < \varepsilon_0/3 \quad \text{and} \quad \|f_\tau\Omega - f_{\tau(\theta)}\Omega\| < \varepsilon_0/3.$$

Then we have

$$\begin{aligned}\varepsilon_0 &\leqslant \|f_\tau^{(n_{k_0})}\Omega - f_\tau\Omega\| \leqslant \|f_\tau^{(n_{k_0})}\Omega - f_{\tau(\theta)}^{(n_{k_0})}\Omega\| + \|f_{\tau(\theta)}^{(n_{k_0})}\Omega - f_{\tau(\theta)}\Omega\| + \|f_{\tau(\theta)}\Omega - f_\tau\Omega\| \\ &\leqslant \varepsilon_0/3 + \sup_t \|f^{(n_{k_0})}(t) - f(t)\|_\infty + \varepsilon_0/3 < \varepsilon_0,\end{aligned}$$

a contradiction. Consequently,

$$\lim_{n\to\infty} f_\tau^{(n)}\Omega = f_\tau\Omega,$$

and since for each $a' \in \mathscr{A}'$

$$\|f_\tau^{(n)}(a'\Omega) - f_\tau(a'\Omega)\| = \|a'[f_\tau^{(n)}\Omega - f_\tau\Omega]\| \leqslant \|a'\|\,\|f_\tau^{(n)}\Omega - f_\tau\Omega\|,$$

the result follows. ∎

3. STOCHASTIC INTEGRAL AS A CLOSABLE OPERATOR

In this section we shall perform a construction of a stochastic integral as a closable operator analogous to that of stopping in the previous section. Let $(f(t))$ and $(g(t))$ be $\mathscr{A}$-valued processes. The definition of stochastic integral involves sums

$$S_\theta^l(g;f) = \sum_{i=1}^{m} [f(t_i)-f(t_{i-1})]\, g(t_{i-1}),$$

$$S_\theta^r(g;f) = \sum_{i=1}^{m} g\,(t_{i-1})[f(t_i)-f(t_{i-1})]$$

for a partition $\theta = \{0 = t_0 < t_1 < \ldots < t_m = +\infty\}$ of $[0, +\infty]$. If a limit of $S_\theta^l(g;f)$ as θ refines exists, it is called the *left stochastic integral* of $(g(t))$ with respect to $(f(t))$, and it is denoted by $\int df(t)\, g(t)$, while a limit of $S_\theta^r(g;f)$ is called the *right stochastic integral* of $(g(t))$ with respect to $(f(t))$, and it is denoted by $\int g(t)\, df(t)$ (see [7] for more details). As for stopping we are interested in the existence of this limit on $\mathscr{A}'\Omega$. It turns out that the following counterpart of Theorem 2.4 holds.

THEOREM 3.1. *Let $(f(t))$ and $(g(t))$ be $\mathscr{A}$-valued processes. Assume that there exists*

$$\lim_\theta S_\theta^l(g;f)\,\Omega = \xi,$$

and define the left stochastic integral by

$$[\textstyle\int df(t)\, g(t)]\,(a'\Omega) = a'\xi, \qquad a' \in \mathscr{A}'.$$

Then $\int df(t)\, g(t)$ is a densely defined closable operator whose closure is affiliated to $\mathscr{A}$. Moreover,

$$\overline{\textstyle\int df(t)\, g(t)} = [\textstyle\int g(t)^*\, df(t)^*]^*,$$

where the integral $\int g(t)^*\, df(t)^*$ is defined as an operator on $\mathscr{A}'\Omega$ in an obvious way.

The proof is essentially the same as that of Theorem 2.4. ∎

In what follows, when referring to the situation described above, we shall use the phrase 'there exists an operator stochastic integral $\int df(t)\, g(t)$'. We also restrict attention to the left integral, the corresponding results for the right one being obvious.

As is well known, there is a duality between stopping and integration. This duality allows us to obtain operator stochastic integrals for a random time with respect to a process which admits operator stopping. Namely, we have

THEOREM 3.2. *Let $(f(t))$ be an $\mathscr{A}$-valued process, and let $\tau = (E_t)$ be a random time. If there exists an operator stopping f_τ, then the operator stochastic integral $\int df(t)\,\tau(t)$ exists. Moreover, the formula*

$$\textstyle\int df(t)\,\tau(t) = f(\infty) - f_\tau$$

holds.

Proof. Let $\theta = \{0 = t_0 < t_1 < \ldots < t_m = +\infty\}$ be a partition of $[0, +\infty]$. Using the Abel transformation (summation by parts), we get

$$S^l_\theta(\tau; f) = \sum_{i=1}^{m} [f(t_i) - f(t_{i-1})] E_{t_{i-1}}$$

$$= f(t_m) E_{t_m} - f(t_0) E_{t_0} - \sum_{i=1}^{m} f(t_i)(E_{t_i} - E_{t_{i-1}}) = f(+\infty) - f_{\tau(\theta)}.$$

The existence of operator stopping for $(f(t))$ yields the existence of $\lim_\theta f_{\tau(\theta)}\Omega$, thus the existence of $\lim_\theta S^l_\theta(\tau; f)\Omega$, and the result follows. ∎

Since from Theorem 2.10 we know that all processes in $\mathscr{U}(\mathscr{P})$ admit operator stopping, we obtain

Corollary 3.3. *Let $f \in \mathscr{U}(\mathscr{P})$, and let $\tau = (E_t)$ be a random time. Then the operator stochastic integral $\int df(t)\tau(t)$ exists, and*

$$\int df(t)\tau(t) = -f_\tau.$$

This follows from the fact that for $f \in \mathscr{U}(\mathscr{P})$ we have $f(+\infty) = 0$. ∎

Taking into account the linearity of operator stochastic integral we may extend the set of integrands as follows:

Theorem 3.4. *Let $f \in \mathscr{U}(\mathscr{P})$, and let g be a simple predictable process which is a finite linear combination of elementary predictable processes. Then the operator stochastic integral $\int df(t) g(t)$ exists.*

Proof. The process g is a linear combination of random times and processes of the form $[0_P]$ with P being a projection in $\mathscr{A}_0$, and

$$[0_P](t) = \begin{cases} P & \text{for } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since the integral of a random time exists, we are concerned only with the existence of the integral $\int df(t)[0_P](t)$. The integral sums $S^l_\theta([0_P]; f)$ are of the form

$$S^l_\theta([0_P]; f) = [f(t_1) - f(0)] P,$$

so the existence of the operator integral $\int df(t)[0_P](t)$ amounts to showing the existence of $\lim_{t\to 0+} f(t) P\Omega$. Now for each elementary predictable process

$$g(t) = \lambda_0 [0_F] + \sum_{k=1}^{r} \lambda_k (\sigma_k, \varrho_k], \qquad \sigma_k = (F^{(k)}_t),\ \varrho_k = (Q^{(k)}_t),$$

the limit $\lim_{t\to 0+} g(t)$ exists in the strong operator topology since

$$\lim_{t\to 0+} [0_F](t) = 0$$

and

$$\lim_{t\to 0+}(\sigma_k, \varrho_k] = \lim_{t\to 0+}(F_t^{(k)}) - \lim_{t\to 0+} Q_t^{(k)},$$

where the limits on the right-hand side clearly exist because $(F_t^{(k)})$ and $(Q_t^{(k)})$ are increasing families of projections. It follows that for each simple predictable process g which is a finite linear combination of elementary predictable processes the limit $\lim_{t\to 0+} g(t)$ exists.

Let now $(f^{(n)})$ be an approximating sequence of processes as above for f. Given $\varepsilon > 0$ we choose n_0 such that

$$\sup_t \|f(t) - f^{(no)}(t)\|_\infty < \varepsilon/3,$$

and $\delta > 0$ such that for $0 < s,\ t < \delta$ we have

$$\|[f^{(no)}(s) - f^{(no)}(t)]\,P\Omega\| < \varepsilon/3.$$

Then

$$\|[f(s)-f(t)]\,P\Omega\| \leqslant \|[f(s)-f^{(no)}(s)]\,P\Omega\| + \|[f^{(no)}(s)-f^{(no)}(t)]\,P\Omega\| \\ + \|[f^{(no)}(t)-f(t)]\,P\Omega\| < \varepsilon/3+\varepsilon/3+\varepsilon/3 = \varepsilon,$$

showing that $\{f(t)P\Omega\}$ is Cauchy at 0, which completes the proof. ∎

Taking into account Theorem 2.11 and the duality between stopping and integration it is not difficult to show the following

THEOREM 3.5. *Let f and g be as in Theorem 3.4, and let $(f^{(n)})$ be an approximating sequence for f of simple predictable processes which are finite linear combinations of elementary predictable processes. Then*

$$\lim_{n\to\infty} \int df^{(n)}(t)\, g(t)(a'\Omega) = \int df(t)\, g(t)(a'\Omega), \qquad a' \in \mathscr{A}'. \quad ∎$$

So far we have obtained integrals with the integrator from the class $\mathscr{U}(\mathscr{P})$ and the integrand being a simple predictable process which is a finite linear combination of elementary predictable processes. In the last part of this section we shall show that the classes of the integrators and the integrands can be interchanged. To this end we start with

PROPOSITION 3.6. *Let $\sigma = (F_t)$ and $\tau = (E_t)$ be random times. Then the operator stochastic integral $\int d\sigma(t)\,\tau(t)$ exists.*

Proof. By Proposition 2.6 there exists an operator stopping σ_τ and the result follows from Theorem 3.2. ∎

Again, by linearity we can extend the above integral to the integrands being elementary predictable processes, if we take into account that

$$\int d\sigma(t)\,[0_P](t) = \lim_{t\to 0+}(F_t - F_0)P = F_{0+}P.$$

By the same token we extend the integral to the integrands being finite linear combinations of elementary predictable processes. To go further we again need a version of the contraction lemma.

LEMMA 3.7. *Let $(f(t))$ be an $\mathscr{A}$-valued process, and let $\sigma = (F_t)$ be a random time. Then for any partition $\theta = \{0 = t_0 < t_1 < \ldots < t_m = +\infty\}$ we have*

$$\|S^l_\theta(f;\sigma)\Omega\| \leqslant \sup_t \|f(t)\|_\infty .$$

Proof. The calculation is essentially the same as that in Lemma 2.9. Namely,

$$\begin{aligned}
\|S^l_\theta(f;\sigma)\Omega\|^2 &= \Big\|\sum_{i=1}^m (F_{t_i}-F_{t_{i-1}}) f(t_{i-1})\Omega\Big\|^2 \\
&= \sum_{i,j=1}^m \langle f(t_{j-1})^*(F_{t_j}-F_{t_{j-1}})(F_{t_i}-F_{t_{i-1}}) f(t_{i-1})\Omega, \Omega\rangle \\
&= \sum_{i=1}^m \langle (F_{t_i}-F_{t_{i-1}}) f(t_{i-1}) f(t_{i-1})^*\Omega, \Omega\rangle \\
&= \sum_{i=1}^m \langle f(t_{i-1}) f(t_{i-1})^*(F_{t_i}-F_{t_{i-1}})\Omega, \Omega\rangle \\
&\leqslant \sum_{i=1}^m \|f(t_{i-1}) f(t_{i-1})^*\|_\infty \langle (F_{t_i}-F_{t_{i-1}})\Omega, \Omega\rangle \\
&\leqslant \sup_t \|f(t_{i-1}) f(t_{i-1})^*\|_\infty \sum_{i=1}^m \langle (F_{t_i}-F_{t_{i-1}})\Omega, \Omega\rangle = \sup_t \|f(t)\|^2_\infty . \blacksquare
\end{aligned}$$

In the next proposition we extend the stochastic integral to processes in $\mathscr{U}(\mathscr{P})$.

PROPOSITION 3.8. *Let $f \in \mathscr{U}(\mathscr{P})$ and let σ be a random time. Then the operator stochastic integral $\int d\sigma(t) f(t)$ exists.*

Proof. It is similar to the proof of Theorem 2.10. For an approximating sequence $(f^{(n)})$ and given $\varepsilon > 0$ we choose n_0 such that

$$\sup_t \|f(t) - f^{(no)}(t)\|_\infty < \varepsilon/4. \tag{3.1}$$

Since we know that

$$\lim_\theta S^l_\theta(f^{(no)};\sigma) = \int d\sigma(t) f^{(no)}(t),$$

we can find a partition θ_0 such that for each partition $\theta \geqslant \theta_0$

$$\Big\|S^l_\theta(f^{(no)};\sigma)\Omega - \int d\sigma(t) f^{(no)}(t)\Omega\Big\| < \varepsilon/4. \tag{3.2}$$

Then for any partitions $\theta', \theta'' \geqslant \theta_0$ we have

$$\|S^l_{\theta'}(f;\sigma)\Omega - S^l_{\theta''}(f;\sigma)\Omega\| \leqslant \|S^l_{\theta'}(f;\sigma)\Omega - S^l_{\theta'}(f^{(no)};\sigma)\Omega\|$$
$$+\|S^l_{\theta'}(f^{(no)};\sigma)\Omega - S^l_{\theta''}(f^{(no)};\sigma)\Omega\| + \|S^l_{\theta''}(f^{(no)};\sigma)\Omega - S^l_{\theta''}(f;\sigma)\Omega\|$$
$$\leqslant \|S^l_{\theta'}(f-f^{(no)};\sigma)\Omega\| + \left\|S^l_{\theta'}(f^{(no)};\sigma)\Omega - \int d\sigma(t) f^{(no)}(t)\Omega\right\|$$
$$+\left\|\int d\sigma(t) f^{(no)}(t)\Omega - S^l_{\theta''}(f^{(no)};\sigma)\Omega\right\| + \|S^l_{\theta''}(f^{(no)}-f;\sigma)\Omega\|.$$

But the first and fourth terms on the right-hand side of the above inequality are by virtue of Lemma 3.7 and (3.1) estimated from above by $\varepsilon/4$, while the second and third terms are by virtue of (3.2) estimated from above also by $\varepsilon/4$, which gives

$$\|S^l_{\theta'}(f;\sigma)\Omega - S^l_{\theta''}(f;\sigma)\Omega\| < \varepsilon \qquad \text{for } \theta', \theta'' \geqslant \theta_0.$$

Consequently, the limit $\lim_\theta S^l_\theta(f;\sigma)\Omega$ exists. Now Theorem 3.1 yields the claim. ■

In our final step we again extend by linearity the integral with respect to a random time to the integral with respect to an elementary predictable process and next to the integral with respect to a simple predictable process which is a finite linear combination of elementary predictable processes, taking into account that

$$\int d[0_P](t) f(t) = -Pf(0).$$

Thus we obtain

THEOREM 3.9. *Let $f \in \mathscr{U}(\mathscr{P})$, and let g be a simple predictable process which is a finite linear combination of elementary predictable processes. Then the operator stochastic integral $\int dg(t) f(t)$ exists.* ■

Again it is not difficult to show that our integral may be obtained by a limiting procedure. Namely, we have

THEOREM 3.10. *Let f and g be as in Theorem 3.9, and let $(f^{(n)})$ be an approximating sequence for f. Then*

$$\lim_{n\to\infty} \int dg(t) f^{(n)}(t)(a'\Omega) = \int dg(t) f(t)(a'\Omega), \qquad a' \in \mathscr{A}'. \ ■$$

Let us finally comment on an important missing fact which would nicely complete the presented theory. We have obtained operator stochastic integrals $\int df(t) g(t)$ and $\int dg(t) f(t)$ for $f \in \mathscr{U}(\mathscr{P})$ and g — simple predictable. It would be desirable to obtain these integrals for both f and g in $\mathscr{U}(\mathscr{P})$. However, it still remains a challenging open problem.

REFERENCES

[1] C. Barnett and V. Camillo, *Stopping and projections of non-adapted processes*, Soochow J. Math. 23 (2) (1997), pp. 187–212.

[2] C. Barnett and T. Lyons, *Stopping non-commutative processes*, Math. Proc. Cambridge Philos. Soc. 99 (1986), pp. 151–161.

[3] C. Barnett, R. F. Streater and I. F. Wilde, *Quasi-free quantum stochastic integrals for the CAR and CCR*, J. Funct. Anal. 52 (1983), pp. 19–57.

[4] C. Barnett and B. Thakrar, *A non-commutative random stopping theorem*, J. Funct. Anal. 88 (1990), pp. 342–350.

[5] C. Barnett and I. F. Wilde, *Random times and time projections*, Proc. Amer. Math. Soc. 110 (1990), pp. 425–440.

[6] C. Barnett and I. F. Wilde, *Random times, predictable processes and stochastic integration in finite von Neumann algebras*, Proc. London Math. Soc. 67 (3) (1993), pp. 355–383.

[7] A. Łuczak and A. A. Mohammed, *Stochastic integration in finite von Neumann algebras*, preprint.

[8] S. Strâtilâ and L. Zsido, *Lectures on von Neumann Algebras*, Abacus Press, Tunbridge Wells 1979.

[9] F. J. Yeadon, *Non-commutative L^p-spaces*, Math. Proc. Cambridge Philos. Soc. 77 (1975), pp. 91–102.

Faculty of Mathematics
University of Łódź
Łódź, Poland

Received on 14.12.2001